All about *twice*

Mike Spivey

A problem in the Functional Programming course concerns a function *twice*, defined by

$$\mathit{twice}\, f\ x = f\ (f\ x).$$

It asks for the value of $\mathit{twice}\ \mathit{twice}\ \mathit{twice}\ f\ x$, but here we generalize this to ask what is the value of $T_n\ f\ x$ where T_n is defined by saying that $T_0 = \mathit{id}$ and $T_{n+1} = T_n\ \mathit{twice}$. The stated problem is the special case $n = 3$.

Let us first introduce a couple of other notations. We will write $f^n\ x$ for the result of applying f to x at total of n times, so $f^0\ x = x$ and $f^{n+1}\ x = f\ (f^n\ x)$. Note that here the brackets associate to the right, and in our definition of T_n they associate to the left, so that

$$\mathit{twice}^3\ f\ x = \mathit{twice}\,(\mathit{twice}\,(\mathit{twice}\, f))\ x = f^8\ x,$$

and this is different from $T_3\ f\ x$.

Let's also adopt Knuth's notation for 'hyperpowers', and write $2 \uparrow\uparrow n$ for the quantity defined by $2 \uparrow\uparrow 0 = 1$ and $2 \uparrow\uparrow (n+1) = 2^{2\uparrow\uparrow n}$. In the symbols of the TEXbook (Exercise 18.45),

$$2 \uparrow\uparrow n = 2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}} \Big\}{\scriptstyle n}.$$

We first observe that $\mathit{twice}^n\ f\ x = f^{2^n}\ x$. This can be proved by induction: for the basis $n = 0$, we see that $\mathit{twice}^0\ f\ x = f\ x = f^{2^0}\ x$; and assuming that the result holds for $n = k$, we can calculate as follows when $n = k + 1$:

$$\begin{aligned}\mathit{twice}^{k+1}\ f\ x &= \mathit{twice}\,(\mathit{twice}^k\ f)\ x \\ &= \mathit{twice}^k\ f\ (\mathit{twice}^k\ f\ x) = f^{2^k}\ (f^{2^k}\ x) = f^{2^{k+1}}\ x.\end{aligned}$$

Now we are in a position to prove that $T_n\ f\ x = f^{2\uparrow\uparrow n}\ x$, again by induction on n. For the basis, take $n = 0$, and both sides simplify to $f\ x$. Now assume the result holds for $n = k$, and put $n = k + 1$. We can apply the induction hypothesis with f replaced by *twice* and x replaced by f, like this:

$$T_{k+1}\ f\ x = T_k\ \mathit{twice}\, f\ x = \mathit{twice}^{2\uparrow\uparrow k}\ f\ x = f^{2^{2\uparrow\uparrow k}}\ x = f^{2\uparrow\uparrow(k+1)}\ x.$$

This completes the proof. As special cases, we obtain $T_3\ f\ x = f^{16}\ x$ and $T_4\ f\ x = f^{65536}\ x$.